JOURNAL OF APPROXIMATION THEORY 36, 86–90 (1982)

Convergence Rates for Monotone Cubic Spline Interpolation

FLORENCIO I. UTRERAS

Departamento de Matemáticas, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 5272, Correo 3, Santiago, Chile

Communicated by Carl de Boor

Received September 25, 1981

Given a monotone function $g \in H^2[0, 1]$ and a sequence of meshes Δ_n such that $\lim_{n \to \infty} |\Delta_n| = 0$, we consider the monotone cubic spline interpolating g at the knots of Δ_n. If we call $\sigma_{M,n}$ this function, we show that

(i) $\lim_{n\to\infty} \int_0^1 (g''(t) - \sigma''_{M,n}(t))^2 \, dt = 0$,

(ii) $\| g^{(k)} - \sigma^{(k)}_{M,n} \|_\infty = o(|\Delta_n|^{3/2-k})$, $k = 0, 1$.

DERIVATION OF THE RESULTS

Let g be a monotone increasing function belonging to $H^2[0, 1] = \{f: [0, 1] \to \mathbb{R}: f, f'$ are absolutely continuous and $f'' \in L^2[0, 1]\}$.

Given a sequence of meshes $\Delta_n = \{0 < t_1^n < t_2^n < \cdots < t_n^n < 1\}$, let us call $\sigma_{M,n}$ the cubic monotone spline interpolator of g at the knots of Δ_n. That is, the unique solution of the problem

$$\int_0^1 (\sigma''_{M,n}(t))^2 \, dt = \operatorname*{Min}_{u \in M_n} \int_0^1 (u''(t))^2 \, dt, \tag{1}$$

where

$$M_n = \{u \in H^2[0, 1] \mid u(t_i^n) = g(t_i^n),\ i = 1,..., n;\ u'(t) \geqslant 0,\ \forall t \in [0, 1]\}. \tag{2}$$

For a proof of the existence and uniqueness of σ_{Mn}, see [2]. We are interested in the behavior of $\sigma_{M,n}$ as n increases; more precisely, we are interested in the convergence properties of $\sigma_{M,n}$ to g.

First, we establish the analog of the first integral relation for natural cubic splines.

0021-9045/82/090086-05$02.00/0

LEMMA 1 (First Integral Relation). *For all $u \in M_n$, we have*

$$\int_0^1 [u''(t) - \sigma''_{M,n}(t)]^2 \, dt \leqslant \int_0^1 [u''(t)]^2 \, dt - \int_0^1 [\sigma''_{M,n}(t)]^2 \, dt. \tag{3}$$

Proof. From the definition of $\sigma_{M,n}$, we conclude that

$$\int_0^1 [\sigma''_{M,n}(t)]^2 \, dt \leqslant \int_0^1 [u''(t)]^2 \, dt \qquad \text{for all} \quad u \in M_n,$$

but M_n is a closed convex set, hence, using the well-known theorem for the projection on a convex set (see [3]), we conclude that

$$\int_0^1 (u''(t) - \sigma''_{M,n}(t)) \, \sigma''_{M,n}(t) \, dt \geqslant 0. \tag{4}$$

By developing the square in the left-hand side of (3), we have

$$\begin{aligned} &\int_0^1 [u''(t) - \sigma''_{M,n}]2 \, dt \\ &\quad = \int_0^1 (u''(t))^2 \, dt - 2 \int_0^1 u''(t) \, \sigma''_{M,n}(t) \, dt + \int_0^1 [\sigma''_{M,n}(t)]^2 \, dt. \end{aligned} \tag{5}$$

But (4) is equivalent to

$$\int_0^1 u''(t) \, \sigma''_{M,n}(t) \, dt \geqslant \int_0^1 [\sigma''_{M,n}(t)]^2 \, dt.$$

Introducing this into (5), we get

$$\begin{aligned} &\int_0^1 (u''(t) - \sigma''_{M,n}(t))^2 \, dt \\ &\quad \leqslant \int_0^1 (u''(t))^2 \, dt - 2 \int_0^1 (\sigma''_{M,n}(t))^2 \, dt + \int_0^1 (\sigma''_{M,n}(t))^2 \, dt. \end{aligned}$$

This concludes the proof. ∎

Let s_n be the natural cubic spline interpolating g at the knots of Δ_n. As is well known (cf. [1]), s_n satisfies

$$\int_0^1 [s''_n(t)]^2 = \operatorname*{Min}_{u \in I_n} \int_0^1 [u''(t)]^2 \, dt, \tag{6}$$

where

$$I_n = \{u \in H^2[0, 1] \mid u(t_i^n) = g(t_i^n),\ i = 1, 2, \dots, n\}. \tag{7}$$

Now we establish a relationship between s_n and $\sigma_{M,n}$.

THEOREM 1. *We have the inequality*

$$\int_0^1 (\sigma''_{M,n}(t) - s''_n(t))^2\, dt \leqslant \int_0^1 [g''(t) - s''_n(t)]^2\, dt - \int_0^1 (g''(t) - \sigma''_{M,n}(t))^2\, dt. \tag{9}$$

Proof. $\sigma_{M,n}$ being an element of I_n, the first integral relation for s_n implies that

$$\begin{aligned}\int_0^1 (\sigma''_{M,n}(t) - s''_n(t))^2\, dt &= \int_0^1 (\sigma''_{M,n}(t))^2\, dt - \int_0^1 (s''_n(t))^2\, dt \\ &= \int_0^1 (\sigma''_{M,n}(t))^2\, dt - \int_0^1 (g''(t))^2\, dt + \int_0^1 (g''(t))^2\, dt - \int_0^1 (s''_n(t))^2\, dt \\ &= -\left[\int_0^1 (g''(t))^2\, dt - \int_0^1 (\sigma''_{M,n}(t))^2\, dt\right] + \int_0^1 (g''(t))^2\, dt - \int_0^1 (s''_n(t))^2\, dt.\end{aligned}$$

Given that $g \in M_n \subseteq I_n$, we can apply the first integral relations for $\sigma_{M,n}$ and s_n to get the desired result. ∎

COROLLARY. *Let $\sigma_{M,n}$ be the monotone cubic spline interpolating g at the knots of ϑ_n. If $g \in H^2[0, 1]$ and $\lim_{n\to\infty} |\vartheta_n| = 0$, where*

$$|\vartheta_n| = \mathrm{Max}(t_1^n, t_2^n - t_1^n, \dots, t_n^n - t_{n-1}^n, 1 - t_n^n\},$$

then

$$\lim_{n\to\infty} \int_0^1 (\sigma''_{M,n}(t) - g''(t))^2\, dt = 0. \tag{9}$$

Proof. Using Theorem 1, we have

$$\begin{aligned}\int_0^1 (\sigma''_{M,n}(t) - s''_n(t))^2\, dt &\leqslant \int_0^1 [g''(t) - s''_n(t)]^2\, dt - \int_0^1 (g''(t) - \sigma''_{M,n}(t))^2\, dt \\ &\leqslant \int_0^1 [g''(t) - s''_n(t)]^2\, dt.\end{aligned}$$

But s_n is the cubic spline, then $|\Delta_n| \to 0$ implies that $\int_0^1 [g''(t) - s_n''(t)]^2 \, dt \to 0$, from where we get the desired result. ∎

As is well known from the theory of spline functions (see [1, 5]), this result will allow us to obtain estimates for the convergence rates. We do this in the following

THEOREM 2. *Let $g \in H^2[0, 1]$ be monotone increasing. Consider a sequence of meshes Δ_n such that $\lim_{n\to\infty} |\Delta_n| = 0$, and call $\sigma_{M,n}$ the monotone cubic spline interpolating g at Δ_n. Then*

$$\| g^{(k)} - \sigma_{M,n}^{(k)} \|_\infty = o(|\Delta_n|^{3/2-k}), \qquad k = 0, 1. \tag{10}$$

Proof. Given that $\sigma_{M,n}$ interpolates g at the knots of Δ_n, we have

$$g(t_i^n) - \sigma_{M,n}(t_i^n) = 0, \qquad i = 1. 2,..., n, \tag{11}$$

Using Rolle's theorem, we deduce the existence of $\xi_1^n,..., \xi_{n-1}^n$ such that

$$g'(\xi_i^n) - \sigma_{M,n}'(\xi_i^n) = 0, \qquad \xi_i^n \in [t_i^n, t_{i+1}^n], \quad i = 1,..., n-1$$

we then have,

$$g'(t) - \sigma_{M,n}'(t) = \int_{\xi_i^n}^t [g''(r) - \sigma_{M,n}''(r)] \, dr, \qquad t \in [\xi_i^n, \xi_{i+1}^n]$$

using now Schwartz inequality we obtain

$$| g'(t) - \sigma_{M,n}'(t)| \leqslant |t - \xi_i^n|^{1/2} \left| \int_{\xi_i^n}^t (g''(r) - \sigma_{M,n}''(r))^2 \, dr \right|^{1/2}. \tag{12}$$

But, it is easy to see that

$$\operatorname{Max}\{\xi_1^n, \xi_2^n - \xi_1^n,..., \xi_{n-1}^n - \xi_{n-2}^n, 1 - \xi_{n-1}^n\} \leqslant 2 |\Delta_n|.$$

Replacing this inequality into (12) we finally obtain

$$| g'(t) - \sigma_{M,n}'(t)| \leqslant (2 |\Delta_n|)^{1/2} \left| \int_0^1 (g''(r) - \sigma_{M,n}''(r))^2 \, dr \right|^{1/2},$$

$$t \in [\xi_i^n, \xi_{i+1}^n]$$

In $[0, \xi_1^n]$, $[\xi_{n-1}^n, 1]$ we proceed in the same way and finally obtain

$$\| g' - \sigma_{M,n}' \|_\infty \leqslant |\Delta_n|^{1/2} \cdot 2^{1/2} \left[\int_0^1 (g''(r) - \sigma_{M,n}''(r))^2 \, dr \right]^{1/2}.$$

Using now the preceding corollary, we get the desired result for $k = 1$.

From this, we obtain the convergence rate for $k=0$, in a standard way,

$$g(t)-\sigma_{M,n}(t)=\int_{t_i^n}^{t}(g'(t)-\sigma'_{M,n}(t))\,dt,$$

hence

$$\begin{aligned}|g(t)-\sigma_{M,n}(t)| &\leqslant |t-t_i^n|\,\|g'-\sigma'_{M,n}\|_\infty\\ &\leqslant |\Delta_n|\,\|g'-\sigma'_{M,n}\|_\infty, \qquad t\in[t_i^n,t_{i+1}^n],\end{aligned}$$

and the same holds for $t\in[0,t_1^n]$, $[t_n^n,1]$. This allow us to conclude that

$$\|g-\sigma_{M,n}\|_\infty\leqslant|\Delta_n|^{3/2}\left[\int_0^1(g''(t)-\sigma''_{M,n}(t))^2\right]^{1/2}\cdot 2^{1/2}.$$

This concludes the proof. ∎

The preceding results tell us that, even with an additional constraint, the rate of convergence of cubic splines is preserved. This remarkable property is not very surprising because it has also been stated for the interpolation of monotone functions by monotone polynomials (see [4]).

It is also well known that natural cubic spline interpolants do not converge at the optimal rate and that the inclusion of boundary conditions can raise the order of convergence up to $o(|\Delta_n|^4)$. We think that it might be possible to do the same for monotone splines, but more information on the nature of $\sigma_{M,n}$ is needed.

Acknowledgments

The author wishes to thank two referees for their helpful comments.

References

1. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, "The Theory of Splines and their Applications," Academic Press, New York, 1967.
2. U. Hornung, Monotone spline interpolation, *in* "Numerische Methoden der Approximationstheorie" (Herausgegeben von L. Collatz, G. Meinardus, und H. Werner), Band 4, Birkhäuser, Basel, 1978.
3. P. J. Laurent, "Approximation et optimisation," Hermann, Paris, 1972.
4. G. Lorentz and K. Zeller, "Degree of approximation by monotone polynomials," *J. Approx. Theory* **1** (1968), 501–504.
5. R. S. Varga, "Functional Analysis and Approximation Theory in Numerical Analysis," SIAM Series, 1971.

Printed in Belgium